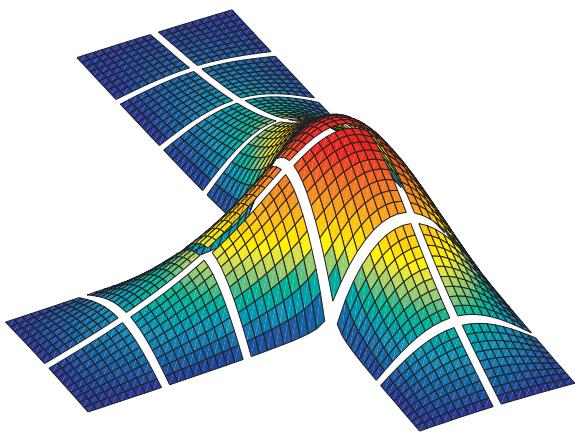


# An optimized Schwarz method in the Jacobi-Davidson method for eigenvalue problems



**Menno Genseberger**

joint work with  
**Gerard Sleijpen &**  
**Henk van der Vorst**



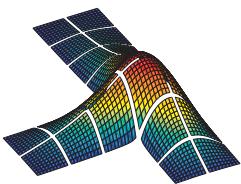
**Utrecht University**



**CWI - Amsterdam**

# outline

- the Jacobi-Davidson method
- an optimized Schwarz method
- enhancement
- tuning coupling
- deflation
- applications
- more details
- questions?



# the Jacobi-Davidson method

(Sleijpen & van der Vorst '96)

iterative method for eigenvalue problems

for standard eigenvalue problem  $\mathbf{A} \mathbf{x} = \lambda \mathbf{x}$  it

- extracts  $(\theta, \mathbf{u}) \approx (\lambda, \mathbf{x})$  from search subspace

construct  $H \equiv \mathbf{V}^* \mathbf{A} \mathbf{V}$ ,

solve  $H s = \theta s$ , and compute  $\mathbf{u} = \mathbf{V} s$

where  $\mathbf{V}$  orthonormal basis for search subspace

- corrects  $\mathbf{u}$

compute correction  $\mathbf{t}$  from *correction equation*:

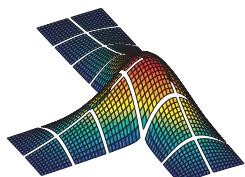
$$\mathbf{t} \perp \mathbf{u}, \quad \mathbf{P} \mathbf{B} \mathbf{P} \mathbf{t} = \mathbf{r}$$

where  $\mathbf{P} \equiv \mathbf{I} - \frac{\mathbf{u} \mathbf{u}^*}{\mathbf{u}^* \mathbf{u}}$ ,  $\mathbf{B} \equiv \mathbf{A} - \theta \mathbf{I}$ , and  $\mathbf{r} \equiv -\mathbf{B} \mathbf{u}$

- expands search subspace with  $\mathbf{t}$

$$\mathbf{V}_{new} = [\mathbf{V} \mid \mathbf{t}^\perp]$$

where  $\mathbf{t}^\perp = \alpha (\mathbf{I} - \mathbf{V} \mathbf{V}^*) \mathbf{t}$  such that  $\|\mathbf{t}^\perp\|_2 = 1$



# an optimized Schwarz method

(Tan & Borsboom '93, generalization of Tang '92)

for linear system  $\mathbf{B} \mathbf{y} = \mathbf{d}$  it

- enhances linear system

$$\mathbf{B} \mathbf{y} = \mathbf{d} \longrightarrow \mathbf{B}_C \tilde{\mathbf{y}} = \underline{\mathbf{d}}$$

- splits  $\mathbf{B}_C = \mathbf{M} - \mathbf{N}$  such that preconditioner  $\mathbf{M}$  invertible locally on subdomains
- computes solution of enhanced system by

Richardson iteration

$$\tilde{\mathbf{y}}^{(i+1)} = \tilde{\mathbf{y}}^{(i)} + \mathbf{M}^{-1} (\underline{\mathbf{d}} - \mathbf{B}_C \tilde{\mathbf{y}}^{(i)})$$

or

a more general Krylov method with

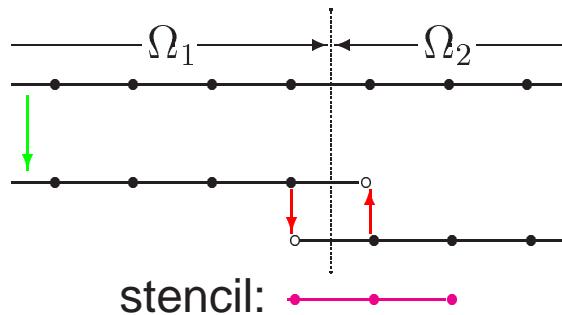
$$\begin{aligned} \mathcal{K}_m \left( \mathbf{M}^{-1} \mathbf{B}_C, \mathbf{M}^{-1} \underline{\mathbf{d}} \right) &= \text{span} \left( \mathbf{M}^{-1} \underline{\mathbf{d}}, \right. \\ &\quad \left. \mathbf{M}^{-1} \mathbf{B}_C \mathbf{M}^{-1} \underline{\mathbf{d}}, \dots, \left( \mathbf{M}^{-1} \mathbf{B}_C \right)^{m-1} \mathbf{M}^{-1} \underline{\mathbf{d}} \right) \end{aligned}$$

- tunes  $C$  for “minimal” spectral radius of  $\mathbf{M}^{-1} \mathbf{N}$   
( $\mathbf{M}^{-1} \mathbf{B}_C = \mathbf{I} - \mathbf{M}^{-1} \mathbf{N} \Rightarrow$  error of Richardson iteration amplified by *error propagation matrix*  $\mathbf{M}^{-1} \mathbf{N}$ )

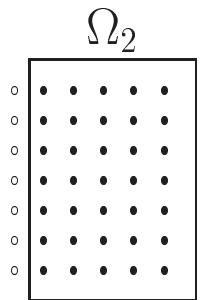
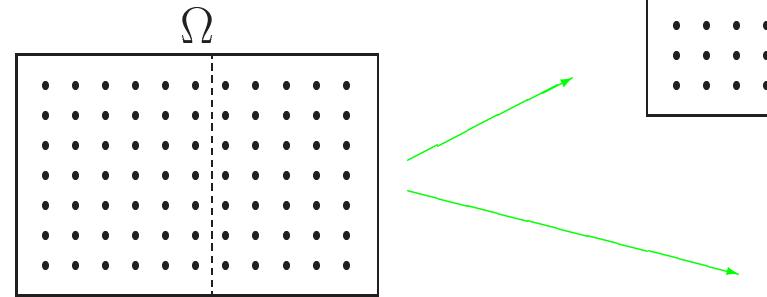


# enhancement

1 dimension



2 dimensions

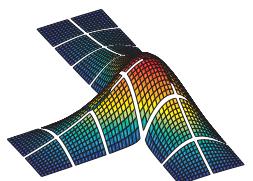
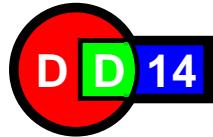


enhancement system

$$\mathbf{B} \mathbf{y} = \mathbf{d} \rightarrow \mathbf{B}_C \mathbf{y} = \underline{\mathbf{d}} :$$

$$\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{1\ell} & \mathbf{B}_{1r} & \mathbf{0} \\ \mathbf{B}_{\ell 1} & \mathbf{B}_{\ell\ell} & \mathbf{B}_{\ell r} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{r\ell} & \mathbf{B}_{rr} & \mathbf{B}_{r2} \\ \mathbf{0} & \mathbf{B}_{2\ell} & \mathbf{B}_{2r} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ y_\ell \\ y_r \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ d_\ell \\ d_r \\ \mathbf{d}_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{1\ell} & \mathbf{B}_{1r} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{\ell 1} & \mathbf{B}_{\ell\ell} & \mathbf{B}_{\ell r} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\ell\ell} & \mathbf{C}_{\ell r} & -\mathbf{C}_{\ell\ell} & -\mathbf{C}_{\ell r} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C}_{r\ell} & -\mathbf{C}_{rr} & \mathbf{C}_{r\ell} & \mathbf{C}_{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{r\ell} & \mathbf{B}_{rr} & \mathbf{B}_{r2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{2\ell} & \mathbf{B}_{2r} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ y_\ell \\ \tilde{y}_r \\ \tilde{y}_\ell \\ y_r \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ d_\ell \\ 0 \\ 0 \\ d_r \\ \mathbf{d}_2 \end{bmatrix}$$



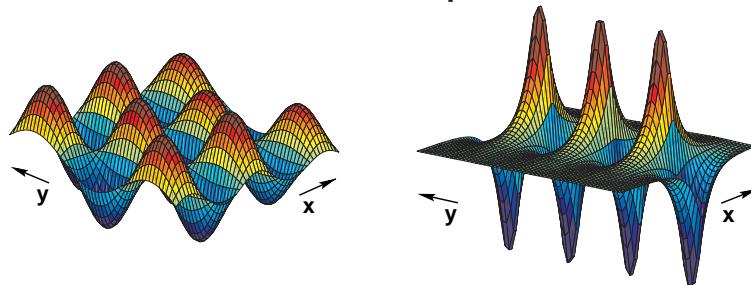
# tuning coupling

analysis spectrum  $\mathbf{M}^{-1} \mathbf{N}$  for model eigenvalue problem:

advection-diffusion operator  $L_x \otimes \mathbf{I} + \mathbf{I} \otimes L_y$ ,  
constant coefficients, two subdomains with  
interface in  $y$ -direction



$x$ -direction: harmonic & exponential behavior



$y$ -direction: coupling by eigenvectors of  $L_y$

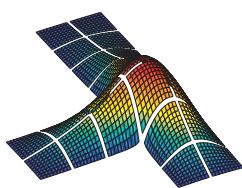
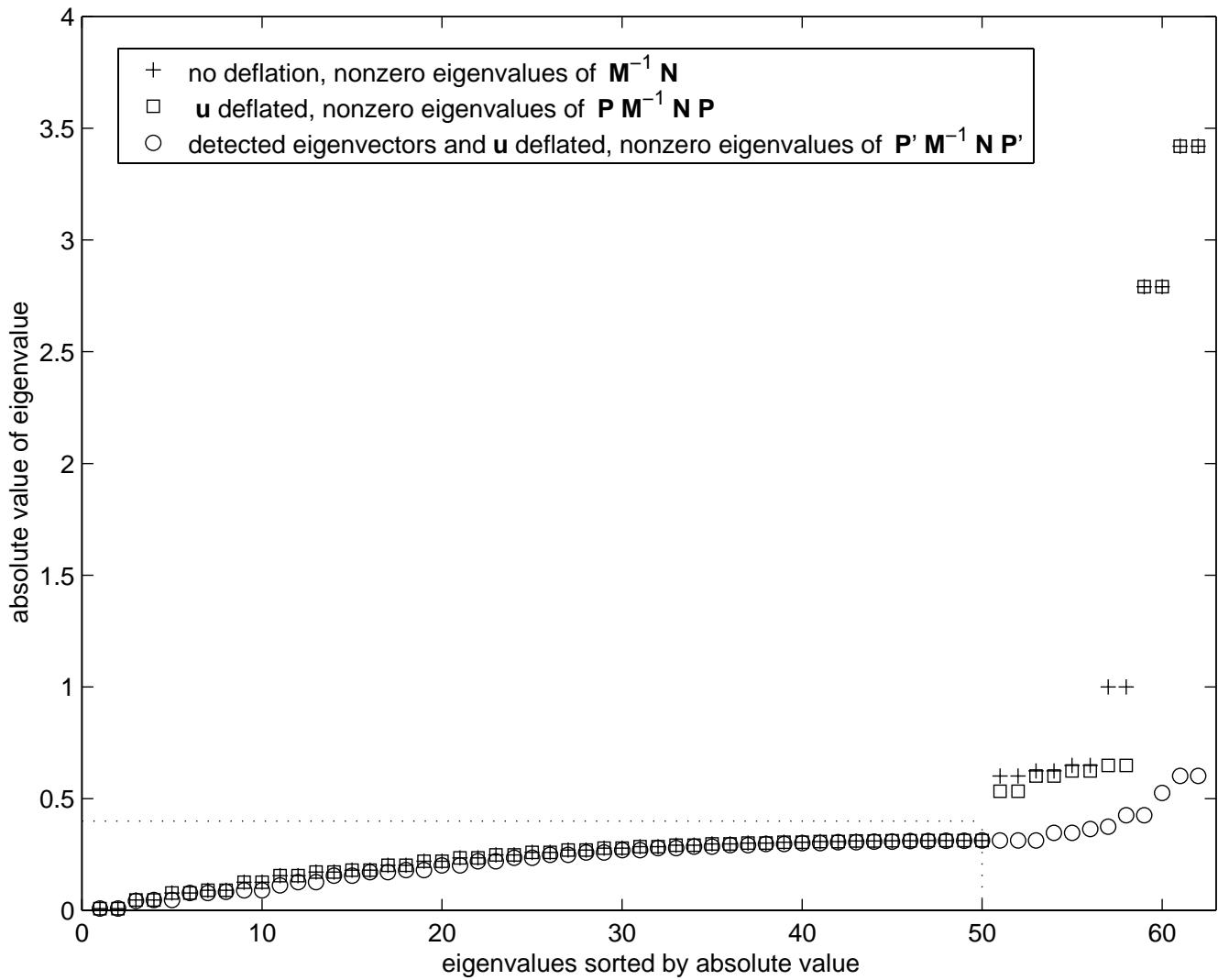


subblocks of  $C$  can be any linear combination  
of powers of  $L_y$ ,  
e.g.  $C_{\ell\ell} = C_{rr} = \mathbf{I}$ ,  $C_{\ell r} = C_{r\ell} = \alpha \mathbf{I} + \beta L_y$ ,  
minimization spectral radius  $\mathbf{M}^{-1} \mathbf{N}$  w.r.t.  $\alpha, \beta, \dots$



# deflation

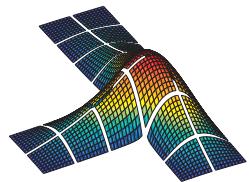
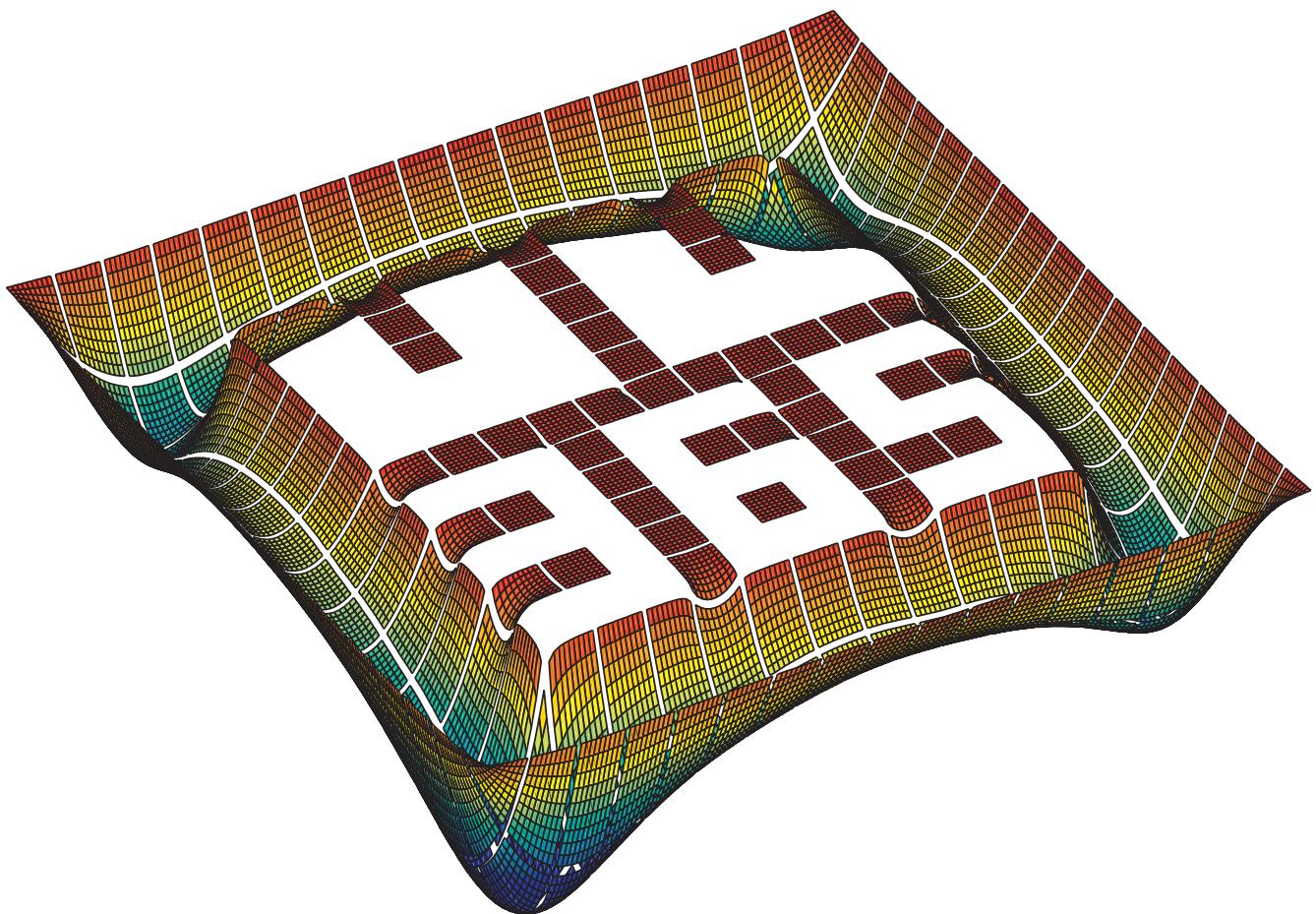
correction equation indefinite  $\rightarrow$  deflation



# applications

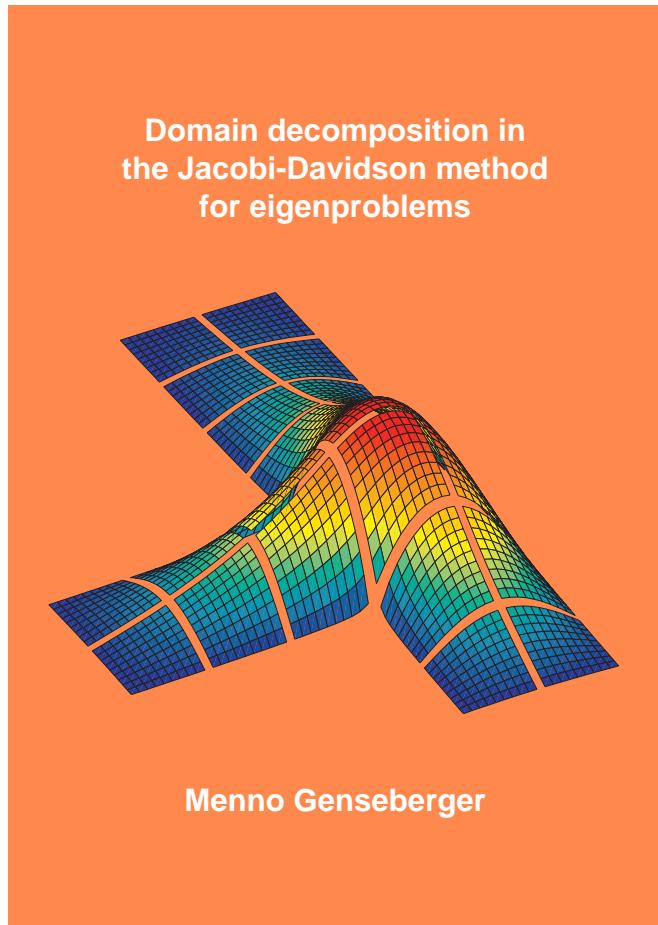
results from analysis yield accurate estimates of optimal coupling  $C$  for

- more than two subdomains
- variable coefficients
- complicated geometries



## more details

### Ph.D. thesis



hardcopy available here  
for electronic version surf to  
[www.cwi.nl/~genseber](http://www.cwi.nl/~genseber)  
see also poster at exhibition

for more information about Jacobi-Davidson check  
[www.math.uu.nl/people/sleijpen](http://www.math.uu.nl/people/sleijpen)  
[www.math.uu.nl/people/vorst](http://www.math.uu.nl/people/vorst)

